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METHOD OF EXTRACTING SINGULARITIES IN THE PROBLEM OF THE HYDROELASTIC VIBRATIONS OF A SHELL EXCITED BY CONCENTRATED FORCES*

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An asymptotic justification for the procedure /1, 2/ of matching the integrals of the vibrations of a shell and the Helmholtz equations for the acoustic pressure is given in the example of the problem of the vibrations of a closed spherical shell in an infinite medium, excited by forces applied at the poles of the shell. The order of constructing the approximate solution, based on replacement of the fluid influence by several apparent masses each of which is related to a specific integral of the shell equations, and on extraction of the singularities of the solution at the point of application of the force, is traced. The results are compared with the exact solution of the problem in the form of series in spherical functions /3/.

1. We will write the original system of equations of the axisymmetric vibrations of a spherical shell and an ideal compressible fluid while separating out the time dependence, given in the form $e^{-i\omega t}$ in the functions of the load Z, the acoustic pressure p and the shell displacements u and w

$$(1+\nu)\, w_{,\theta} - \Phi_{,\theta} - (1-\nu)\,\mu_0 u = 0,\quad \Phi = \sin^{-1}\theta\,(u \sin\theta)_\theta, \tag{1.1}$$
$$[\nabla^4 - (1-\nu)\nabla^2]\, w - c_*^2 \left[\frac{\Phi}{1-\nu} - \left(\frac{2}{1-\nu} - \alpha_0^2\right) w\right] = \frac{r_0^4}{D}(Z - p|_S)$$
$$\nabla^2 p + (r^2 p_{,r})_{,r} + (kr)^2 p = 0,\quad \lim_{r\to\infty} r\,(p_{,r} - ikp) = 0$$
$$p_{,r}|_S = \omega^2 \rho w,\quad \alpha_0 = \omega r_0 (\rho_0/E)^{1/2},\quad k = \omega/c$$
$$\mu_0 = 1 + \alpha_0^2 (1+\nu),\quad D = 2Eh^3/[3(1-\nu^2)]$$
$$c_*^2 = 2Ehr_0^2/D,\quad \nabla^2 = (\ )_{,\theta\theta} + \operatorname{ctg}\theta\,(\ )_{,\theta},\quad (\ )_{,x} = \partial/\partial x$$

Here r and θ are spherical coordinates ($r = r_0$ is the equation of the shell surface S), h is half the shell thickness, ω is the angular frequency of the vibrations, ρ_0, E, ν are the density, Young's modulus, and Poisson's ratio of the shell material, and ρ and c are the density and velocity of sound in the fluid.

One of the effective approximate methods of solving two-dimensional problems of the type (1.1) is to reduce them to a one-dimensional problem on the shell surface by using an exponential representation of the fluid pressure integrals in the neighbourhood of the shell

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/1, 2/. Setting

$$p(r, \theta) = p(r_0, \theta) \exp[-a(r - r_0)], \ \mathrm{Re}\, a > 0 \tag{1.2}$$

where a is a previously unknown damping factor, and substituting this representation into the condition of non-penetration (1.1), we obtain a relation between the fluid pressure functions on the shell surface and the deflection

$$p(r_0, \theta) = -\omega^2 \rho a^{-1} w(\theta) \tag{1.3}$$

by means of which the function $p(r_0, \theta)$ is eliminated from the equations of the shell vibrations. The original problem is thereby reduced to constructing the solution of a system of equations of the vibrations of a shell with certain attached masses.

The simple exponential representation (1.2) has an asymptotic justification. We will examine the exact solution of the Helmholtz equation for the acoustic pressure in a direction normal to the shell. For a number n of half-waves along the meridian this solution has the form of a spherical Hankel function $h_n^{(1)}(z)$, $z = kr$. Depending on the relationship between the argument z and the index n, two kinds of asymptotic forms $h_n^{(1)}(z)$ are possible in the neighbourhood of the shell. For $n \geqslant z_0$, $z_0 = kr_0$ the behaviour of $h_n^{(1)}(z)$ at any distance from the shell is described by the uniform asymptotic exapansion /4/

$$h_n^{(1)}(z) = \sqrt{{}^1/_2 \pi / z}\, H_q^{(1)}(qz_1), \quad q = n + {}^1/_2, \ qz_1 = z \tag{1.4}$$

$$H_q^{(1)}(qz_1) = 2e^{-\pi i/3} [4\xi(1 - z_1^2)^{-1}]^{1/4} q^{-1/3} \mathrm{Ai}(t_0)\, [1 + O(q^{-1})]$$

$$\frac{2}{3}\xi^{3/2} = \int_{z_1}^{1} \frac{\sqrt{1 - t^2}}{t}\, dt, \ z_1 \leqslant 1; \quad \frac{2}{3}(-\xi)^{3/2} = \int_{1}^{z_1} \frac{\sqrt{t^2 - 1}}{t}\, dt, \quad z_1 \geqslant 1$$

$$t_0 = e^{2\pi i/3} q^{2/3} \xi$$

The zero of the argument of the Airy function $\mathrm{Ai}(t_0)$ in this expansion determines the location of the spherical transfer surface in the space occupied by the fluid /5/. Between the shell and the transfer surface the real and imaginary part of the functions

$$2e^{-\pi i/3}\, \mathrm{Ai}(t_0) = \mathrm{Ai}(t) - i\, \mathrm{Bi}(t), \ t = q^{2/3}\xi$$

vary according to a law close to the exponential law

$$\mathrm{Ai}(t) = {}^1/_2 \pi^{-1/2} t^{-1/4} \exp(-\xi_1)\, (1 + O(q^{-1})) \tag{1.5}$$

$$\mathrm{Bi}(t) = \pi^{-1/2} t^{-1/4} \exp(\xi_1)\, (1 + O(q^{-1})); \quad \xi_1 = q \int_{z_1}^{1} \frac{\sqrt{1 - x^2}}{x}\, dx$$

They oscillate with slow damping outside the limits of the transfer surface.

Substituting the function $h_n^{(1)}(z)$ into system (1.1), we obtain the following inertial term

$$\frac{r_0^4}{D}\, p\,|_S = c_*^2 \alpha_0^2 \mu w, \quad \mu = -\frac{\rho}{z_0} \frac{h_n^{(1)}(z_0)}{h_n^{(1)\prime}(z_0)}, \quad g = \frac{r_0}{2h} \frac{\rho}{\rho_0} \tag{1.6}$$

at the site $p|_S$.

Taking account of the asymptotic representations (1.4) and (1.5), we represent the approximate expression for the fluid associated mass coefficient (AMC) in the form

$$\mu_a = g\, [{}^1/_2 + \sqrt{q^2 - z_0^2}\, (1 - i\varepsilon)]^{-1}, \ \varepsilon = \exp(-2\xi_0) \tag{1.7}$$

$$\xi_0 = \xi_1 |_{z = z_0}$$

(it is taken into account that the inequality $\exp(-2\xi_0) \leqslant 4 \exp(2\xi_0))$ is satisfied at some appreciable distance of the transfer surface from the shell). It is hence seen that the acoustic pressure component damped out exponentially deep in the medium introduces the predominant contribution to the magnitude of the AMC for large variability of the solution in the meridian direction.

When the inequality $q \gg z_0$ is conserved the asymptotic representation (1.7) yields the correct estimate of the AMC even for small values of the frequency parameter z_0 down to zero. Indeed, for small z_0 and $n > z_0$ the exact value of the AMC in (1.6) tends to the analogous value for an incompressible fluid $\mu = g(1 + n)^{-1} [1 + O(z_0^2)]$. Letting z_0 in (1.7) also tend to zero, we obtain that $\xi_0 \to \infty$, $\varepsilon \to 0$ and $\mu_a \to g(1 + n)^{-1}$, i.e., $\mu_a \approx \mu$ even as $z_0 \to 0$.

In the case when $z_0 \gg n$ there is no transfer surface in the fluid. Consequently, the wave asymptotic form of the function $h_n^{(1)}(z)$ is set up directly on the shell surface. Using a Hankel binomial asymptotic expansion ((9.2.7) and (9.2.13) from /4/) for the functions $H_q^{(1)}(z)$ and $H_q^{(1)\prime}(z)$, we arrive at the following formula for the AMC:

$$\mu = g(1 - iz_0)^{-1} \tag{1.8}$$

This formula can also be continued in the low frequency range with the inequality $z_0 > n$ conserved. In fact, as $n \to 0$

$$-zh_n^{(1)\prime}(z)/h_n^{(1)}(z) \to zh_1^{(1)}(z)/h_0^{(1)}(z) = 1 - iz$$

Therefore, (1.7) and (1.8) yield asymptotic representations for the AMC that are uniform in the frequency and variability of the solutions for two characteristic combinations between the variability parameters n and the frequency z_0. Integrals of primarily bending, rapidly-varying vibrations modes correspond to the inequality $n > z_0$, and slowly varying integrals with a predominance of the tangential shell displacement vector components correspond to the condition $n < z_0$ * (* Golovanov, V.A., Muzychenko, V.V. Peker, F.N. and Popov A.L. The scattering and radiation of sound by elastic shells in a fluid. Preprint No. 261, Inst. Problem Mekhaniki Akad. Nauk SSSR, Moscow, 1985.)

2. We will use the representations obtained for the AMC to construct the integrals for the original system (1.1).

We introduce the characteristic index $s^2 = q(q+1)$ (q is not necessarily an integer) of the system (1.1) on the shell surface. For this we use the resolving equation

$$\Delta^2 w + s^2 w = 0 \tag{2.1}$$

into which the Helmholtz equation transfers on the contact surface when $p(r_0, \theta)$ is replaced by $w(\theta)$ in the form (1.3). Assuming the characteristic index s^2 to be identical for both the shell displacement vector components and for the fluid pressure on its surface, we arrive at the characteristic equation

$$\begin{gathered}(1+\nu)(1-\nu)^{-1}c_*^2 s^2 - [s^2 - (1-\nu)\mu_0][s^2(s^2 + 1 - \nu) + \\ c_*^2(2(1-\nu)^{-1} - \alpha_*^2)] = 0 \\ \alpha_*^2 = \alpha_0^2(1+\mu)\end{gathered} \tag{2.2}$$

in which the quantity μ is related to s^2 by (1.7) or (1.8) (for $\mu = 0$ Eq.(2.2) is identical with the known characteristic equation in the case of spherical shell vibrations in a vacuum /6/).

We will use the asymptotic AMC representation in the form (1.7) to determine the characteristic indices of rapidly varying integrals. A complex transcendental equation in s^2 is obtained on substituting (1.7) into (2.2). Let us consider the asymptotic procedure to determine the roots of this equation by using the smallness of ε.

We first take $\varepsilon = 0$, which corresponds to the initial exponential representation (1.2) in which only the acoustic pressure components being damped from the shell are taken into account. Expressing s^2 in (2.2) in terms of a (in the principal approximation $s^2 = a^2$), we arrive at a characteristic equation of the seventh degree

$$\beta_0 + \beta_1 s + \ldots + \beta_7 s^7 = 0 \tag{2.3}$$

whose coefficients are obviously expressed in terms of the coefficients of (2.2), where some of them (for the lowest powers of s) contains the large parameter c_*^2. Certain roots of (2.3) are "large" compared with the other roots and the rapidly varying integrals of system (1.1) correspond to it.

Analysis of the roots of (2.3) using the Routh-Hurwitz criterion shows that three out of the five large roots have a positive real part that satisfies the condition of damping of the integrals (1.2) with distance from the shell. One of them (s_1) is real and positive, the other two ($s_{2,3}$) are complex conjugates with positive real part. Oscillating integrals of the resolving Eq.(2.1) correspond to the root s_1, and integrals of the edge-effect type that damp out with oscillations in the neighbourhood of points of application of the force correspond to the roots $s_{2,3}$.

The value of the root s_1 obtained from the equation of the seventh degree should be considered as the initial approximation to the real root ($s_1 = s_1^{(0)}$). Newton's method

$$s_1^{(k+1)} = s_1^{(k)} - f/(df/ds)\,|_{s=s_1^{(k)}},\ k = 0, 1, 2 \ldots \tag{2.4}$$

is used later to solve the equation $f(s_1, \mu(s_1)) = 0$, where $f(s_1, \mu(s_1))$ is the left-hand side of (2.2) while $\mu = gs_1^{-1}(1 + i\varepsilon(s_1))$.

The first step in this iteration process can be written in explicit form. Let μ_0 be the AMC value corresponding to $\varepsilon = 0$ (i.e., the initial appoximation for s_1). Let us set

$\mu^{(1)} = \mu^{(0)}(1 + i\varepsilon)$ and, respectively, $s_1^{(1)} = s_1^{(0)}(1 + i\delta)$, where δ is an unknown small parameter and let us substitute these expressions into (2.2). Retaining first-order infinitesimal terms, we obtain a relation between ε and δ

$$\delta = \frac{\mu^{(0)}\Omega^2\varepsilon}{s_1^{(0)}} \frac{s_1^{(0)} - 1 + \nu - \Omega^2}{s_1^{(0)}(3s_1^{(0)} - 8 - 2\Omega^2)c_*^{-2} + 1 - \nu^2 - \Omega^2(1 + \mu^{(0)})}, \quad \Omega^2 = \alpha_0^2(1 - \nu^2) \tag{2.5}$$

Carrying out the refining procedure for the characteristic indices of edge effect type integrals corresponding to the complex roots $a_{2,3}$, $\operatorname{Re} a_{2,3} > 0$ has no value in principle since these indices possess large imaginary parts commensurate with the real parts in the original approximation $(\varepsilon = 0)$.

Small roots of (2.3) are not taken into account when constructing the rapidly varying integrals. The characteristic indices for the slowing varying integrals are determined directly from (2.2) since the AMC (1.7) does not contain unknown parameters. Because integrals of this kind can be constructed from a membrane system /7/, we obtain an explicit expression for the corresponding root of (2.2) (denoted by s_4^2)

$$s_4^2 = \mu_0 \frac{2 - \alpha_*^2(1 - \nu)}{1 - \alpha_*^2}, \quad \alpha_*^2 = \alpha_0^2\left(1 + \frac{g}{1 - iz_0}\right)$$

3. We now turn to the construction of the integrals of the resolving system. Two linearly independent integrals of the resolving Eq.(2.1) correspond to each of the values of the characteristic indices s_j^2 $(j = 1, \ldots, 7)$. The Legendre functions $P_q(\pm t)$, $t = \cos\theta$ /8/ can be selected by such integrals for non-integer indices q $(s^2 = q(q + 1))$.

We will represent the general integrals of system (1.1) for the functions of the deflection $w(\theta)$, meridian displacement $u(\theta)$, and fluid pressure on the shell surface $p(r_0, \theta)$ in the form of sums of rapidly and slowly varying integrals

$$w(\theta) = \sum_{j=1}^{N} F_j(\theta), \quad u(\theta) = \sum_{j=1}^{N} f_j F_j'(\theta) \tag{3.1}$$

$$p(r_0, \theta) = -\Omega_0^4 \sum_{j=1}^{N} \mu_j F_j(\theta), \quad \Omega_0^4 = 2\omega^2\rho_0 h$$

$$F_j(\theta) = c_j^+ P_{q_j}(t) + c_j^- P_{q_j}(-t), \quad f_j = (1 + \nu)[(1 - \nu)\mu_0 - s_j^2]^{-1}$$

where $c_j^\pm$ $(j = 1, \ldots, N)$ are arbitrary constants, s_j^2 are roots of the characteristic Eq.(2.2), and μ_j are the AMC.

We will set up the necessary number (N) of integrals $P_{q_j}(\pm t)$ $(j = 1, \ldots, N)$ by starting from the requirement that the general solutions for $w(\theta)$ and $u(\theta)$ have regular singularities at the points of concentrated application of the force. The order of the principal singularity of the fluid pressure function on the shell surface should not, as follows from (1.1) and (1.3), exceed the analogous index for the shell deflection function.

The main singularities of shell displacement functions under static application of concentrated forces are classified in /9/. The inertial components (shell mass and associated mass of the fluid) do not increase the order of the main singularity of these functions. For a given method of external load application the main singularity of the dynamic deflection function of a shell making contact with a fluid is the same at the upper pole as for a plate under axisymmetric loading $\varkappa(r_0\theta)^2 \ln\theta$, $\varkappa = Q/(8\pi D)$ (Q is the amplitude of the force), the function $u(\theta)$ has no singularities at the shell poles.

The expression for the main singularity of the function $p|_S$ at the point of application of the force can also be estimated by starting from the solution of the problem about the action of a periodic force on an infinite plate lying on a liquid half-space /10/. Omitting the intermediate calculations, we obtain the following expression for the main singularity $p|_S$

$$\varkappa_1(r_0\theta)^5 \ln\theta, \quad \varkappa_1 = -\varkappa D^{-1}(\omega^2\rho/192)^2$$

Comparing it with the corresponding formula for the shell deflection function shows that the order of the singularity of the fluid pressure function is significantly lower.

It follows from expansions of the Legendre functions in the neighbourhood of $t = \pm 1$ /8/ that they have logarithmic singularities at the shell poles ($P_q(-t)$ at the upper pole and $P_q(t)$ at the lower pole) of higher order than there should be for solutions for the shell displacement and fluid pressure at the points of application of the force. We require the satisfaction of the following conditions to separate the regular singularities in the solution (2.5): the coefficient of $\ln\theta$ and $\ln(\pi - \theta)$ must be equal in the expansions of $w(\theta)$ and $p(r_0, \theta)$, as must the coefficient of $(r_0\theta)^2 \ln\theta$ (and for $r_0^2(\pi - \theta)^2 \ln(\pi - \theta)$) respectively) to a known value of $\varkappa$ in the expansion of $w(\theta)$. The coefficients for θ^{-1} $(\theta \to 0)$ and $(\pi - \theta)^{-1}(\theta \to$

π) should vanish in expression $u(\theta)$.

Taking into account the order of the singularity for the pressure that is lower than for the deflection, satisfaction of the additional conditions that the coefficients of $(r_0\theta)^n \ln\theta$ and $r_0^n(\pi-\theta)^n\ln(\pi-\theta)$, $n = 2, 4, 6$ in the expansion of the pressure function should be zero should also be required. However, this necessitates taking into account additional integrals that can be obtained when taking acount of large complex-conjugate roots $s_{5,6}$ with negative real parts for the characteristic Eq.(2.3). Integrals of the Helmholtz equation that damp out with distance from the shell can also be set in correspondence with these roots, as follows from the properties of the Legendre function: $P_q(z) = P_{-q-1}(z)$ /4/.

Since the approximate relationship $s_j = a_j = \sqrt{q_j(q_j+1)} \approx q_j$ holds for large roots s_j, where a_j is one of the acoustic pressure damping factors deep in the fluid, two pressure damping factors q_5 and $-q_5-1$ will correspond to an identical integral of the meridian solution (the fifth, say). Consequently, despite the fact that $\operatorname{Re} q_5 < 0$, the index $a_5 = -q_5-1$ ensures the damping nature of the representation (1.2) for the pressure integral corresponding to the fifth shell integral.

Therefore, to extract singularities of the shell displacement vector components, integrals corresponding to the roots $s_1, \ldots, s_4$ are sufficient. To extract the singularities of the pressure function the roots $s_{5,6}$ ($\operatorname{Re} s_{5,6} < 0$) are used in addition, to which the Helmholtz equation integrals that damp with distance from the shell also correspond (note that the appearance of additional linearly independent integrals P_{-q_5-1}, P_{-q_6-1} is due exclusively to the presence of terms in the characteristic equation corresponding to the influence of the fluid). Consequently, a system of twelve algebraic equations is obtained for determining the constants $c_j^{\pm}$ ($j = 1, \ldots, 6$).

The solution (2.5) will be symmetric (antisymmetric) in $w(\theta)$ and $p(r_0, \theta)$ with respect to the equator for identical force amplitudes at the upper and lower poles and coincident of opposite loading phases, while the constants will satisfy the equalities $c_j^+ = \pm c_j{}^* = \pm c_j$, $j = 1, \ldots, 6$. In these cases the determination of the constants reduces to solving a system of six equations.

When a shell is excited by one concentrated force applied to the upper pole, say, expressions (2.5) for the elastic displacements and fluid pressure on the shell surface remain unchanged, but the formulas for $F_j(\theta)$ are written without the functions $P_{q_j}(t)$ and the number of constants is reduced to six.

4. The approximate solution constructed was compared with the known exact solution in a series in spherical functions /3/. Expansions of the deflection, and pressure functions on the surface in the case of shell excitation by two in-phase forces concentrated at the poles have the form (summation over $k = 0, 2, 4, \ldots$)

$$w(\theta) = \frac{1-\nu^2}{2\pi}\frac{Q}{Eh}\sum \frac{(k+{}^1/_2)P_k(\cos\theta)}{b\xi_k - \Delta_k}, \tag{4.1}$$

$$p(r_0, \theta) = \frac{bQ}{\pi r_0^2}\sum \frac{(k+{}^1/_2)P_k(\cos\theta)}{b - \xi_k\Delta_k}$$

$$b = \frac{\rho_0 g r_0 \omega c}{E(1-\nu^2)}, \quad \xi_k = \frac{h_k^{(1)}(z_0)}{h_k^{(1)\prime}(z_0)},$$

$$\Delta_k = (1-\nu^2)\left[nc_*^{-2}(1-\nu-n)+\alpha_0^2\right] - 2(1+\nu) + n\,\frac{(1-\nu^2)c_*^{-2}n + 1 + \nu}{n - 1 + \nu - \alpha_0^2(1-\nu^2)},$$
$$n = k(k+1)$$

There is a small coefficient c_*^{-2} for the term in Δ_k, that is proportional to n^2 and ensures convergence in the terms of the series (4.1). Consequently, it would be necessary to retain a large number of terms (up to 2000) in the series for a numerical realization of the representation (4.1). Confirmation of the accuracy of the computations performed by using (4.1) is obtained by substituting these expressions into the Kirchhoff integral formula for the fluid pressure on the shell surface

$$p(M_0) = \frac{1}{2\pi}\int\limits_S \left[\rho\omega^2 w(M)\frac{e^{ikr}}{r} - p(M)\frac{\partial}{\partial n}\left(\frac{e^{ikr}}{r}\right)\right] dS_M$$
$$r = |M - M_0|$$

In view of the fact that the integrand has a weak singularity at $M = M_0$, quadrature formulas are used that are obtained by replacing the integration variables θ, φ by the introduction of a system of polar coordinates with centre at the singular point M_0. Since the Jacobian of such a transformation tends to zero as the running point M tends to M_0 at the same rate as the distance between M_0 and M, the singularity is eliminated. Modifying a quadrature formula, of Gauss type, say, by using this replacement of the node and weight, we obtain a formula suitable for evaluating the Kirchhoff integral at the point M_0.

Computations showed that the residual between the Kirchhoff integral and the expansion (4.1) of the pressure function is not more than 3.5% in the magnitude of the maximum amplitude at the non-resonance frequency along the whole arc of the meridian.

The exact and approximate solutions were compared according to the magnitudes of the resonance frequencies and the amplitude-frequency dependences (AFD) for the deflection and fluid pressure functions at the lower pole during concentrated force loading at the upper pole. In this case the Legendre functions $P_{qj}(\cos\theta)$ do not occur in the expression for the solution while the functions $P_{qj}(-\cos\theta)$ equal unity at the lower pole; consequently, the error of the approximate solution can be estimated "in pure form".

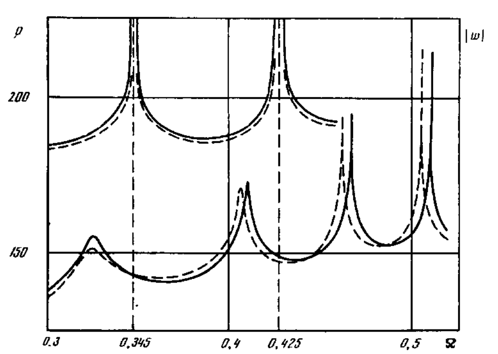
The following shell and fluid parameters were taken: $h = 0.5$ cm, $r_0 = 100$ cm, $\nu = 0.33$, $\rho_0 = 7.85$ g/cm^3, $E = 2.06\cdot10^{11}$ Pa, $\rho = 1$ g/cm^3, $c = 1530$ m/s. The amplitude of the exciting force Q was taken equal to 980 N.

Computations were performed for both compressible and incompressible fluids. In the case of the incompressible fluid the method of extracting singularities enables the exact values of the AMC and the resonance frequencies to be determined since the characteristic indices $s^2 = q(q+1)$ are associated with the AMC by the relationships $\mu = g(1+q)^{-1}$.

n	$\Omega_s\times10^3$	$\Omega_e\times10^3$	$\Omega_t\times10^3$	μ_t	$\Omega_a\times10^3$	μ_a
2	689	345	326	$4.92+0.17i$	324	$5.13+0.11i$
3	818	425	409	$3.53+0.019i$	405	$3.64+0.007i$
4	871	478	466	$2.73+0.001i$	461	$2.83+0.0003i$
5	898	519	510	2.24	504	2.32

Resonances values of the parameter $\Omega = \alpha_0(1-\nu^2)^{1/2}$ and the apparent mass coefficient μ are presented in the table. For a dry shell $\Omega = \Omega_s$, while for a shell in an incompressible fluid $\Omega = \Omega_e$. The values of $\Omega_t, \Omega_a, \mu_t, \mu_a$ are computed for a compresible fluid: Ω_t, μ_t are obtained from the exact solution (4.1) while Ω_a and μ_a are obtained by the approximate method. The number of half-waves n along the shell meridian is indicated in the first column.

The accuracy achieved in de ermining the resonance frequencies is ensured by not more than six iterations of the process (2.4) (for the first frequency). As the frequency rises, the requisite number of iterations is reduced sharply. Note that if we confine ourselves to the step (2.5), then the greatest error in determining the resonance frequency will increase insignificantly: by less than 1.2% for the first resonance.



A fragment of the AFD is represented in the upper part of the figure for the deflection at a shell lower pole in an incompressible fluid that enclose the band of the first resonance frequencies. It is seen that the solution in series (the dashed line) and the solution by the method of extracting singularities (the solid line) are practically identical. In the lower part of the figure we show the AFD for the pressure at the lower pole of a shell submerged in a compressible fluid. The value of Ω, is plotted along the abscissa axis and the pressure level in decibels along the ordinate axis. The solid line corresponds here to the series solution, while the dashes correspond to the solution by the method of singularity extraction. The AFD for the approximate and exact deflection functions have a form similar to that presented.

The testing carried out enables an analgous approximate approach to be used to solving problems of the vibrations of closed shells of revolution with arbitrary meridian outline in a fluid when excited by concentrated loads, since the method of singularity extraction remains the same, in principle, while the asymptotic solution of the Helmholtz equation constructed using Airy functions, as is done in /5/, say, can be used to construct the asymptotic forms of the fluid associated mass.

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REFINED MEMBRANE THEORY OF ELECTROELASTIC SHELLS*

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An analysis of the membrane electroelastic state and the determination of the first vibration eigenfrequencies are often of particular interest in the analysis of thin-walled elements. It is shown how the error of membrane theory can be reduced considerably by introducing certain additional terms into the membrane boundary conditions.

1. To be specific, we will examine piezoceramic shells with thickness polarization. We will write the equations of the theory of the bending of piezoelectric shells to an accuracy of quantities of the order of $(\eta^1 + \eta^{2-2t})$, where t is the index of variability of the fundamental electroelastic state, and η is a small parameter equal to the ratio of half the shell thickness h and its characteristic dimension R:

The equations of equilibrium:

$$\frac{1}{A_i}\frac{\partial T_i}{\partial \alpha_i} + \frac{1}{A_j}\frac{\partial S}{\partial \alpha_j} + k_j(T_i - T_j) + 2k_iS - \frac{p}{R_i}N_i + 2h\rho\omega^2 u_i + X_i = 0 \tag{1.1}$$

$$\sum_{i=1}^{2}\left(\frac{T_i}{R_i} + p\frac{1}{A_i}\frac{\partial N_i}{\partial \alpha_i} + pk_jN_i\right) + 2h\rho\omega^2 w + Z = 0$$

$$N_i = \frac{1}{A_i}\frac{\partial G_i}{\partial \alpha_i} - \frac{1}{A_j}\frac{\partial H_{ij}}{\partial \alpha_j} + k_j(G_i - G_j) - k_i(H_{ij} + H_{ji}) \tag{1.2}$$

$$(k_i = (A_iA_j)^{-1}\,\partial A_i/\partial \alpha_j)$$

(the quantity p in (1.1) should be assumed equal to one; it is required later);

the electroelasticity relations:

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